Roughening and inclination of competition interfaces

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We study the *competition interface* between two clusters growing over a random vacant sector of the plane in a simple setup which allows us to perform formal computations and obtain analytical solutions. We demonstrate that a phase transition occurs for the asymptotic inclination of this interface when the final macroscopic shape goes from curved to noncurved. In the first case it is *random* while in the second one it is deterministic. We also show that the flat case (stationary growth) is a critical point for the fluctuations: for curved and flat final profiles the fluctuations are in the Kardar-Parisi-Zhang (KPZ) scale (2/3); for noncurve final profile the fluctuations are in the same scale of the fluctuations of the initial conditions, which in our model are Gaussian (1/2).

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I. INTRODUCTION

The behavior of the interface of a growing material has been investigated using the Eden model [1], ballistic deposition, and other random systems. Typically, the growing region converges to an asymptotic deterministic shape and its fluctuations depend on the geometry of the initial condition [2,3]. A less well studied phenomenon is the competing growth of two materials. The interface between two growing clusters (*competition interface*) presents a *random* direction on the *same* scale as the deterministic shape [4–6]. In this paper we describe quite explicitly this phenomenon in a simple model. On grounds of universality, this will provide a guide to understanding the interplay between the asymptotics of the competition interface and the final macroscopic shape in models with different growth and competition mechanisms.

We determine the inclination of the competition interface for a growth model called "last passage percolation" in a *random* sector of the plane of angle θ. The growth interfaces are mapped into particle configurations of the *totally asymmetric simple exclusion process* in one dimension (TASEP) [7]. Under Euler space-time rescaling, the particle density of the TASEP converges to a solution of the Burgers equation. This equation has traveling wave solutions (shocks) corresponding to the case $\theta > 180°$, and rarefaction fronts corresponding to $\theta < 180°$. A perturbation at one site of the initial particle configuration (called a *second class particle*) follows a characteristic of the equation or the path of a shock. To establish our results, we map the competition interface linearly onto the path of the second class particle.

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II. THE GROWTH MODEL

The random sector is parametrized by the asymptotic slope of its sides. Let $\lambda \in (0,1]$ and $\rho \in [0,1)$ and define a random path $\gamma_0 = [\gamma_0(j)]_{j\in\mathbb{Z}} \subseteq \mathbb{Z}^2$ with $\gamma_1(0) = (1,0)$, $\gamma_0(0) = (1,1)$, $\gamma_{-1}(0) = (0,1)$ as follows. Starting from $(0,1)$, walk one unit up with probability λ and one unit left with probability $1-\lambda$, repeatedly, to obtain $\gamma_0^1 = [\gamma_0(j)]_{j<0}$. Then, starting from $(1,0)$ walk down with probability ρ and right with probability $1-\rho$ to get $\gamma_0^2 = [\gamma_0(j)]_{j>0}$. γ_0^1 has asymptotic orientation $(\lambda-1,\lambda)$ while γ_0^2 has asymptotic orientation $(1-\rho,-\rho)$. Let C_0 be the sector with boundary γ_0, containing the first quadrant; its asymptotic angle $\theta = \theta_{\lambda,\rho} \in [90°,270°)$ is the angle between $(\lambda-1,\lambda)$ and $(1-\rho,-\rho)$. Notice that $\theta \in [90°,180°)$ if and only if $\rho < \lambda$.

The path γ_0 is the growth interface at time 0. The dynamics are then defined as follows. For each $z \in C_0$ and each $t \geqslant 0$, we have a label $\sigma_t(z) \in \{0,1,2\}$. The label is 0 if z is unoccupied at time t, and is 1 or 2 if z belongs to cluster 1 or 2, respectively. Once occupied, a site remains occupied and keeps the same value forever. Initially, set $\sigma_0(z) = 1$ for all $z \in \gamma_0^1$, $\sigma_0(z) = 2$ for all $z \in \gamma_0^2$ and $\sigma_0(z) = 0$ for all $z \in C_0 \setminus \gamma_0$. Independently each vacant site $z \in C_0 \setminus \gamma_0$ becomes occupied with rate 1 provided $z-(1,0)$ and $z-(0,1)$ are occupied. Let $G(z)$ be the time at which site z becomes occupied. At this time $\sigma_t(z)$ assumes the value $\sigma_t(\bar{z})$ where $\bar{z}$ is the argument that maximizes $G[z-(1,0)]$ and $G[z-(0,1)]$. Thus when a site becomes occupied it joins the cluster of whichever of its two neighbors (below and to the left) became occupied more recently. The label of the site $(1,1)$ may be left ambiguous, but we stipulate that site $(1,2)$ always joins cluster 1, and site $(2,1)$ always joins cluster 2.

The process $(\mathbf{G}_t^1, \mathbf{G}_t^2)$, where $\mathbf{G}_t^k$ is the set of sites $z \in C_0$ such that $\sigma_t(z) = k$, describes the competing spatial growth model. The growth interface at time t is the polygonal path γ_t

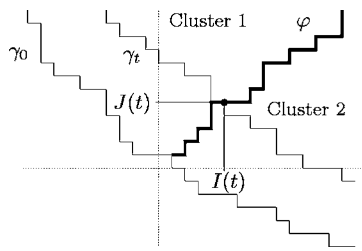

FIG. 1. Growth and competition interfaces.

composed of sites $z \in C_0$ such that $G(z) \leqslant t$ and $G[z+(1,1)] > t$. The *competition interface* $\varphi = (\varphi_n)_{\mathbb{N}}$ is defined by $\varphi_0 = (1,1)$ and, for $n \geqslant 0$, $\varphi_{n+1} = \varphi_n + (1,0)$ if $\varphi_n + (1,1) \in \mathbf{G}^1_\infty$ and $\varphi_{n+1} = \varphi_n + (0,1)$ if $\varphi_n + (1,1) \in \mathbf{G}^2_\infty$. Note that φ chooses locally the shorter step to go up or right, so that φ_{n+1} is the argument that minimizes $\{G[\varphi_n + (1,0)], G[\varphi_n + (0,1)]\}$. This competition interface represents the boundary between those sites which join cluster 1 and those joining cluster 2 (see Fig. 1). The process $\psi(t) = [I(t), J(t)]$ defined by $\psi(t) = \varphi_n$ for $t \in [G(\varphi_n), G(\varphi_{n+1})]$ gives the position of the last intersecting point between the competition interface φ and the growth interface γ_t.

In [8] we prove that with probability one

$$\lim_{n\to\infty} \frac{\varphi_n}{|\varphi_n|} = e^{i\alpha}, \tag{1}$$

where $\alpha \in [0, 90°]$ is given by

$$\tan \alpha = \begin{cases} \dfrac{\lambda\rho}{(1-\lambda)(1-\rho)} & \text{if } \rho \geqslant \lambda \\ \left(\dfrac{1-U}{1+U}\right)^2 & \text{if } \rho < \lambda, \end{cases} \tag{2}$$

and U is a random variable uniformly distributed in $[1-2\lambda, 1-2\rho]$.

III. SIMPLE EXCLUSION AND SECOND-CLASS PARTICLES

The totally asymmetric simple exclusion process $(\eta_t, t \geqslant 0)$ is a Markov process in the state space $\{0,1\}^{\mathbb{Z}}$ whose elements are particle configurations. $\eta_t(j) = 1$ indicates a particle at site j at time t, otherwise $\eta_t(j) = 0$ (a hole is at site j at time t). With rate 1, if there is a particle at site j, it attempts to jump to site $j+1$; if there is a hole at $j+1$ the jump occurs, otherwise nothing happens. The basic coupling between two exclusion processes with initial configurations η_0 and η'_0 is the joint realization (η_t, η'_t) obtained by using the same potential jump times at each site for the two different initial conditions. Let η_0 and η'_0 be configurations of particles differing only at site $X(0) = 0$. With the basic coupling, the configurations at time t differ only at a single site $X(t)$, the position of a so-called *second-class particle*. Such a particle jumps one step to its right to an empty site with rate 1, and jumps backwards one step with rate 1 when a (first class) particle jumps over it.

If η_0 is distributed according to the Bernoulli product measure with density λ for $j \leqslant 0$ and ρ for $j > 0$, then the asymptotic behavior of $X(t)$ shows a phase transition in the line $\lambda = \rho$: with probability one

$$\lim_{t\to\infty} \frac{X(t)}{t} = \begin{cases} 1-\rho-\lambda & \text{if } \lambda \leqslant \rho \\ U & \text{if } \lambda > \rho, \end{cases} \tag{3}$$

where U is a random variable uniformly distributed in $[1-2\lambda, 1-2\rho]$ ([9–11] for the deterministic case and [8,12–14] for the random case).

The limits (3) are based on the following hydrodynamic limits. If η_0 is distributed with the product measure with densities λ and ρ as before, then the macroscopic density evolution is governed by the Burgers equation

$$\lim_{\epsilon\to 0} \epsilon \sum_{x\in\mathbb{Z}} f(x\epsilon)\, \eta_{t/\epsilon}(x) = \int_{\mathbb{R}} f(r) u(r,t)\, \mathrm{d}r \tag{4}$$

with probability one for all $f: \mathbb{R} \to \mathbb{R}$ with compact support, where $u(r,t)$ is the solution of the Burgers equation

$$\frac{\partial u(r,t)}{\partial t} + \frac{\partial}{\partial r}\{u(r,t)[1-u(r,t)]\} = 0, \quad r \in \mathbb{R},\ t \geqslant 0$$

with initial condition $u(r,0) = \lambda$ for $r \leqslant 0$ and ρ for $r > 0$. If $\lambda = \rho$ the solution is constant, if $\lambda < \rho$ it is a *shock*

$$u(r,t) = \begin{cases} \lambda & \text{if } r \leqslant (1-\lambda-\rho)t \\ \rho & \text{if } r > (1-\lambda-\rho)t, \end{cases} \tag{5}$$

and it is a *rarefaction front* if $\lambda > \rho$

$$u(r,t) = \begin{cases} \lambda & \text{if } r \leqslant (1-2\lambda)t \\ \frac{1}{2} - \frac{r}{2(\lambda-\rho)} & \text{if } (1-2\lambda)t < r \leqslant (1-2\rho)t \\ \rho & \text{if } r > (1-2\rho)t \end{cases} \tag{6}$$

([7,15] for initial product measures and [11,16] for initial measures satisfying (4) with $t = 0$; also [17,18]).

The characteristics $v(a,t)$, corresponding to the Burgers equation and emanating from a, are the solutions of $dv/dt = 1 - 2u(v,t)$ with $v(0) = a$. The solutions are constant along the characteristics. When two characteristics carrying a different solution meet, they give rise to a shock. There is only one characteristic emanating from locations where the initial data is locally constant and there are infinitely many characteristics when there is a decreasing discontinuity. In particular, if the initial condition is $u(r,0) = \lambda$ for $r < 0$ and $u(r,0) = \rho$ for $r \geqslant 0$, then the characteristics $v_r(t)$ emanating from the point $v_r(0) = r$ are given by $v_r(t) = r + (1-2\lambda)t$ if $r < 0$ and $v_r(t) = r + (1-2\rho)t$ if $r > 0$. For $r = 0$ there are two cases. When $\lambda \leqslant \rho$, the characteristics emanating from positive sites are slower than those emanating from negative sites. They collide, giving rise to a shock (5) traveling at speed $1-\lambda-\rho$. When $\lambda > \rho$ there are infinitely many characteristics emanating from the origin: for each $s \in [1-2\lambda, 1-2\rho]$ the line $v_0(t) = st$ is a characteristic emanating from 0. The limits (3) show that the second-class particle follows the characteristic when there is only one (that is, when $\lambda = \rho$), that it follows the shock when the initial condition has an increasing discontinuity and that it chooses uniformly one of the characteristics emanating from a decreasing discontinuity.

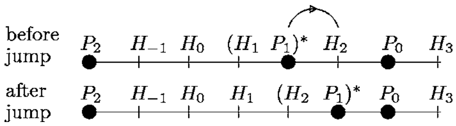

FIG. 2. Pair representation of second class particle.

IV. GROWTH AND SIMPLE EXCLUSION

Rost [7] relates the simple exclusion process to the growth model as follows. Consider initial configurations η_0 for the exclusion process in which $\eta_0(0)=0$ and $\eta_0(1)=1$. Elsewhere let η_0 be distributed according to the Bernoulli product measure with density λ for $j<0$ and ρ for $j>1$. Define the initial growth interface γ_0 by $\gamma_0(0)=(1,1)$ and $\gamma_0(j)-\gamma_0(j-1)=[1-\eta_0(j),-\eta_0(j)]$; then γ_0 has the same distribution as before. Label the particles sequentially from right to left and the holes from left to right, with the convention that the particle at site 1 and the hole at site 0 are both labeled 1. Let $P_j(0)$ and $H_j(0)$, $j\in\mathbb{Z}$ be the positions of the particles and holes, respectively, at time 0. The position at time t of the jth particle $P_j(t)$ and the ith hole $H_i(t)$ are functions of the variables $G(z)$ with $z\in C_0\setminus\gamma_0$ (defined earlier for the growth model) by the following rule: at time $G[(i,j)]$, the jth particle and the ith hole interchange positions. Disregarding labels and defining $\eta_t[P_j(t)]=1$, $\eta_t[H_j(t)]=0$, $j\in\mathbb{Z}$, the process η_t indeed realizes the exclusion dynamics. At time t the particle configuration η_t and the growth interface γ_t still satisfy the same relation as η_0 and γ_0. This connection yields the following *shape theorem* for the growth model. Almost surely

$$\lim_{t\to\infty}\frac{\gamma_t}{t}=\{(r,s)\in\mathbb{R}^2:s=h(r)\},\tag{7}$$

where $h(r)=h_{\lambda,\rho}(r)$ is related to the hydrodynamic limit (5) and (6) by $h'(r)=u(r,1)/[1-u(r,1)]$.

V. SECOND CLASS PARTICLES AND COMPETITION INTERFACES

A key tool in proving (1) and (2) is the observation [13] that the process given by the difference of the coordinates of the competition interface $I(t)-J(t)$ behaves exactly as the second class particle initially put at the origin. To see this call the particle at site 1 *particle and the hole at site 0 *hole, and call this couple *pair. The dynamics of the *pair is the following: it jumps to the right when the *particle jumps to the right, and it jumps to the left when a particle jumps from the left onto the *hole. The *pair then behaves as a second class particle. The only difference is that it occupies two sites while the second class particle occupies only one. The labels of the *particle and *hole change with time. At time 0 they both have label 1 and the labels of the *pair are represented by the point $\varphi_0=(1,1)$, the initial value of the competition interface. If, say, $G(2,1)<G(1,2)$, then the *particle jumps over the second hole before the second particle jumps over the *hole (see Fig. 2). In this case, the labels of the *pair at time $G(2,1)$ are $(2,1)$, which is exactly the argument that minimizes $\{G(2,1),G(1,2)\}$; thus, after the first jump of the *pair, its labels are given by φ_1. By recurrence, φ_n gives exactly the labels of the *pair after its nth jump. Therefore the labels of the *particle and *hole are $J(t)$ and $I(t)$, respectively. In addition, $J(t)-1$ is exactly the number of jumps that the *pair has made backwards up to time t, and $I(t)-1$ is the number of its jumps forwards. This shows that if the exclusion and the growth process are realized in the same space, $X(t)=I(t)-J(t)$. As a consequence of this and (3) we get the following behavior for $\psi(t)$ that implies (1) and (2). Almost surely

$$\lim_{t\to\infty}\frac{\psi(t)}{t}=\begin{cases}((1-\rho)(1-\lambda),\lambda\rho) & \text{if } \lambda\leqslant\rho\\ \frac{1}{4}((U+1)^2,(U-1)^2) & \text{if } \lambda>\rho,\end{cases}\tag{8}$$

where U is a random variable uniformly distributed in $[1-2\lambda,1-2\rho]$.

To prove (8) for $\lambda>\rho$ recall that $P_1(t)$ is the position of the first particle at time t. Thus $J(t)$ is the number of particles that at time zero were to the left of $P_1(0)=1$ and at time t are to the right of $X(t)$. Therefore, $J(t)$ is equal to the number of particles between $X(t)$ and $P_1(t)$ at time t. By the law of large numbers, $P_1(t)/t$ converges to $1-\rho$ and, by (3), $X(t)/t$ converges to U. Hence $J(t)/t$ converges to the integral of the solution of the Burgers equation at time 1 [$u(r,1)$ given by (6)] in the interval $(U,1-\rho)$. Taking $f(r)=\mathbb{1}_{[r\in[U,1-\rho]]}$ in (4)

$$\frac{J(t)}{t}=\frac{1}{t}\sum_{j=X(t)}^{P_1(t)}\eta_t(j)-\to\int_U^{1-\rho}u(r,1)dr=\frac{1}{4}(1-U)^2.$$

Analogously, since $I(t)$ is the number of holes to the right of $H_1(0)=0$ at time zero and to the left of $X(t)$ at time t and $H_1(t)/t$ converges to $-\lambda$ almost surely, we obtain (8) for $\lambda>\rho$. For $\lambda\leqslant\rho$ the same argument works by substituting U above by $1-\lambda-\rho$, the limit position of the second class particle in this case, and taking the solution $u(r,1)$ given by (5).

VI. FLUCTUATIONS

For $\theta>180°$ the second class particle has Gaussian fluctuations produced by the initial profile [19]. This together with the relation above implies that under a diffusive scaling $[I(t),J(t)]$ converges to a bidimensional Gaussian distribution with a nondiagonal covariance matrix computed explicitly [8].

To understand the fluctuations for $\theta\leqslant 180°$ we relate the models to a directed polymer model. For each $z\in C_0\setminus\gamma_0$ let $w_z=G(z)-\max\{G[z-(1,0)],G[z-(0,1)]\}$. Then $(w_z,z\in C_0\setminus\gamma_0)$ is a sequence of i.i.d random variables with an exponential distribution of mean 1. Let $\Pi(z,z')$ be the set of all directed polymers (or up-right paths) $(z_1,\ldots,z_n)$ connecting z to z', and let $G(z',z)$ be the maximum over all $\pi\in\Pi(z',z)$ of $t(\pi)$, the sum of w_z along the polymer π. Each site z has energy $-w_z$, and the polymer π has energy $-t(\pi)$. Thus $-G(z',z)$ is the minimal energy, or ground state, between z' and z. There exists a unique polymer $M(z',z)$ in $\Pi(z',z)$ that attains the maximum. We say that the semi-infinite polymer $(z_n)_{\mathbb{N}}$ is *maximizing* if for all $n<m$ we have $(z_n,\ldots,z_m)=M(z_n,z_m)$. Every semi-infinite maximizing polymer $(z_n)_{\mathbb{N}}$ has an asymptotic inclination $e^{i\alpha}$ [13]. In the competition model, $G(z)=G(\gamma_0,z)$ and for $k=1,2$, $\mathbf{G}_\infty^k$ is the set of sites z such that $M(\gamma_0,z)$ originates from γ_0^k. Denote by ξ the roughening exponent of semi-infinite maximizing polymers.

For $\theta=180°$ $(\lambda=\rho)$ the process is stationary and the connection is explicit. Running the process forward and backward we extend $G(z)$ to all $z\in\mathbb{Z}^2$; $G^{+}=[G(z),z\in\mathbb{Z}^2]$ and $G^{-}=[-G(-z),z\in\mathbb{Z}^2]$ are identically distributed. We define the *forward* competition interface starting at z, $\varphi^z=(\varphi_n^z)_{\mathbb{N}}$, by setting $\varphi_0^z=z$ and putting φ_{n+1}^z equal to the argument of the minimum between $G[\varphi_n^z+(1,0)]$ and $G[\varphi_n^z+(0,1)]$, and the *backward* semi-infinite polymer starting at z, $M^z=(M_n^z)_{\mathbb{N}}$, by setting $M_0^z=z$ and putting M_{n+1}^z equal to the argument of the maximum between $G[M_n^z-(1,0)]$ and $G[M_n^z-(0,1)]$. Note that $\varphi=\varphi^{(1,1)}$ and that M^z is a semi-infinite maximizing polymer. Together with the duality relation $\varphi^z(G^+)=M^z(G^-)$, this shows that the forward competition interface has the same law as the backward semi-infinite maximizing polymer and, in particular, they have the same fluctuations, so that $\chi=\xi$.

For $\theta<180°$ $(\lambda>\rho)$ the competition interface φ is enclosed by two semi-infinite maximizing polymers M^1 and M^2 starting from γ_0^1 and γ_0^2, respectively, and with the same inclination [8]. Therefore $\chi\leqslant\xi$ in this case.

VII. CONCLUSIONS

The connections studied above between the competition interface, the second class particle and maximal polymers fit into the interplay between the fluctuation statistics and the global geometry of the growth interface developed by Prahofer and Spohn [3]. If the final macroscopic profile is curved then the competition interface follows a random direction (characteristic) intersecting the final surface at a point with nonzero curvature. In this case we have the Kardar-Parisi-Zhang (KPZ) scaling and the competition interface gets the transversal fluctuations, indicating the exponent $\chi=2/3$. If the macroscopic profile is not curved we have two different situations. In the flat case (stationary growth) the competition interface also follows the characteristics of the associated hydrodynamic PDE and we still have the KPZ scaling. In the shock case the competition interface gets the longitudinal fluctuations which, in this case, are produced by the Gaussian fluctuations $(\chi=1/2)$ of the initial profile. On microscopic grounds one might suggest different rules for growth and competition. By universality we expect that from the knowledge of the curvature of the final macroscopic shape one can infer the asymptotics of the competition interface. This fits with the exponents founded by Derrida and Dickman [4] in the Eden context since, in this case, the macroscopic profile is curved for angles $\theta>180°$ (we notice that in their simulations they have considered periodic initial conditions and so the longitudinal fluctuations in the shock direction are governed by the exponent $1/3$ [3]).

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[1] M. Eden, *A Two-Dimensional Growth Process*, Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probability, IV 223–239 (University of California Press, Berkeley, CA, 1961).
[2] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[3] M. Prahofer and H. Spohn, Phys. Rev. Lett. **84**, 4882 (2005).
[4] B. Derrida and R. Dickman, J. Phys. A **24**, L191 (1991).
[5] L. P. R. Pimentel, Ph.D. thesis, IMPA, 2004 (unpublished).
[6] Y. Saito and H. Müller-Krumbhaar, Phys. Rev. Lett. **74**, 4325 (1995).
[7] H. Rost, Z. Wahrscheinlichkeitstheor. Verwandte Geb. **58** (1), 41 (1981).
[8] P. A. Ferrari, J. B. Martin, and L. P. R. Pimentel (unpublished).
[9] P. A. Ferrari, *Probability and Phase Transition*, NATO ASI Series C, Vol. 420 (Kluwer, Dordrecht, 1994), p. 35.
[10] F. Rezakhanlou, Ann. Inst. Henri Poincare, Anal. Non Lineaire **12** (2), 119 (1995).
[11] T. Seppäläinen, Trans. Am. Math. Soc. **353**, 4801 (2001).
[12] P. A. Ferrari and C. Kipnis, Ann. I.H.P. Probab. Stat. **31** (1), 143 (1995).
[13] P. A. Ferrari and L. P. R. Pimentel, Ann. Probab. **33** (4), 1235 (2005).
[14] T. Mountford and H. Guiol, Ann. Appl. Probab. **15** (2), 1227 (2005).
[15] F. Rezakhanlou, Commun. Math. Phys. **140** (3), 417 (1991).
[16] T. Seppäläinen, Markov Processes Relat. Fields **4** (4), 593 (1998).
[17] E. Andjel, P. A. Ferrari, and A. Siqueira, Stochastic Proc. Appl. **1132** (2), 217 (2000).
[18] A. Benassi and J.-P. Fouque, Ann. Probab. **15** (2), 546 (1987).
[19] P. A. Ferrari and L. R. Fontes, Probab. Theory Relat. Fields **99**, 305 (1994).